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LETTER TO THE EDITOR

An Ising model in a magnetic field with a boundary

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Abstract. We obtain the diagonal reflection matrices for a recently introduced family of dilute A_L lattice models in which the A_3 model can be viewed as an Ising model in a magnetic field. We calculate the surface free energy from the crossing-unitarity relation and thus directly obtain the critical magnetic surface exponent δ_s for L odd and surface specific heat exponent for L even in each of the various regimes. For $L = 3$ in the appropriate regime we obtain the Ising exponent $\delta_s = -\frac{15}{7}$, which is the first determination of this exponent without the use of scaling relations.

There has been a recent growth of interest in the interaction-round-a-face (IRF) formulation of lattice models in statistical mechanics in the presence of a boundary [1–6]. IRF models such as the restricted solid-on-solid (RSOS) A_L models [7] and the dilute RSOS A_L models [8, 9] are particularly attractive as their solution in terms of elliptic functions correspond to off-critical extensions in which the elliptic nome p measures the deviation from the critical point $p = 0$. We refer to these models here as the A_L models [7] and the dilute A_L models [8]. In the A_L models p is temperature-like while for the dilute RSOS models p is temperature-like for L even but is magnetic-like for L odd. In particular, the A_3 model, in the appropriate regime, can be viewed as a critical Ising model in a thermal field. On the other hand, the dilute A_3 model, in the appropriate regime, can be viewed as a critical Ising model in a magnetic field [8]. The singular part of the free energy of the A_3 model yields the bulk Ising specific heat exponent $\alpha_b = 0$ [7]. Consideration of the surface free energy yields the known Ising surface specific heat exponent [10, 11] $\alpha_s = 1$ [5]. On the other hand, the dilute A_3 model provided a direct calculation of the bulk Ising magnetic exponent $\delta_b = 15$, first from the singular behaviour of the free energy [8] and later from a calculation of the order parameters [12].

In this letter we consider the dilute A_L models with open boundaries and derive the surface free energy from which we obtain the magnetic Ising surface exponent $\delta_s = -\frac{15}{7}$ in the appropriate regime of the A_3 model. This is the first direct calculation of this quantity without the use of scaling relations.

The dilute A_L lattice models [8, 9] are RSOS models with L heights built on the A_L Dynkin diagram with a loop at each node. The non-zero face weights of the off-critical

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dilute A_L models satisfy the star–triangle relation [13] and are given by [8]

$$
\begin{aligned}
W\left(\begin{matrix} a & a \\ a & a \end{matrix}\,\middle|\, u\right) &= \frac{\vartheta_1(6\lambda-u)\vartheta_1(3\lambda+u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)} \\
&\quad -\left(\frac{S(a+1)}{S(a)}\frac{\vartheta_4(2a\lambda-5\lambda)}{\vartheta_4(2a\lambda+\lambda)}\right. \\
&\quad +\left.\frac{S(a-1)}{S(a)}\frac{\vartheta_4(2a\lambda+5\lambda)}{\vartheta_4(2a\lambda-\lambda)}\right)\frac{\vartheta_1(u)\vartheta_1(3\lambda-u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)} \\
W\left(\begin{matrix} a\pm1 & a \\ a & a \end{matrix}\,\middle|\, u\right) &= W\left(\begin{matrix} a & a \\ a & a\pm1 \end{matrix}\,\middle|\, u\right) = \frac{\vartheta_1(3\lambda-u)\vartheta_4(\pm2a\lambda+\lambda-u)}{\vartheta_1(3\lambda)\vartheta_4(\pm2a\lambda+\lambda)} \\
W\left(\begin{matrix} a & a \\ a\pm1 & a \end{matrix}\,\middle|\, u\right) &= W\left(\begin{matrix} a & a\pm1 \\ a & a \end{matrix}\,\middle|\, u\right) \\
&= \left(\frac{S(a\pm1)}{S(a)}\right)^{1/2}\frac{\vartheta_1(u)\vartheta_4(\pm2a\lambda-2\lambda+u)}{\vartheta_1(3\lambda)\vartheta_4(\pm2a\lambda+\lambda)} \\
W\left(\begin{matrix} a & a\pm1 \\ a & a\pm1 \end{matrix}\,\middle|\, u\right) &= W\left(\begin{matrix} a\pm1 & a\pm1 \\ a & a \end{matrix}\,\middle|\, u\right) \\
&= \left(\frac{\vartheta_4(\pm2a\lambda+3\lambda)\vartheta_4(\pm2a\lambda-\lambda)}{\vartheta_4^2(\pm2a\lambda+\lambda)}\right)^{1/2}\frac{\vartheta_1(u)\vartheta_1(3\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \\
W\left(\begin{matrix} a\pm1 & a \\ a & a\mp1 \end{matrix}\,\middle|\, u\right) &= \frac{\vartheta_1(2\lambda-u)\vartheta_1(3\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \\
W\left(\begin{matrix} a & a\mp1 \\ a\pm1 & a \end{matrix}\,\middle|\, u\right) &= -\left(\frac{S(a-1)S(a+1)}{S^2(a)}\right)^{1/2}\frac{\vartheta_1(u)\vartheta_1(\lambda-u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \\
W\left(\begin{matrix} a & a\pm1 \\ a\pm1 & a \end{matrix}\,\middle|\, u\right) &= \frac{\vartheta_1(3\lambda-u)\vartheta_1(\pm4a\lambda+2\lambda+u)}{\vartheta_1(3\lambda)\vartheta_1(\pm4a\lambda+2\lambda)} \\
&\quad +\frac{S(a\pm1)}{S(a)}\frac{\vartheta_1(u)\vartheta_1(\pm4a\lambda-\lambda+u)}{\vartheta_1(3\lambda)\vartheta_1(\pm4a\lambda+2\lambda)} \\
&= \frac{\vartheta_1(3\lambda+u)\vartheta_1(\pm4a\lambda-4\lambda+u)}{\vartheta_1(3\lambda)\vartheta_1(\pm4a\lambda-4\lambda)} \\
&\quad +\left(\frac{S(a\mp1)}{S(a)}\frac{\vartheta_1(4\lambda)}{\vartheta_1(2\lambda)}-\frac{\vartheta_4(\pm2a\lambda-5\lambda)}{\vartheta_4(\pm2a\lambda+\lambda)}\right)\frac{\vartheta_1(u)\vartheta_1(\pm4a\lambda-\lambda+u)}{\vartheta_1(3\lambda)\vartheta_1(\pm4a\lambda-4\lambda)}\,.
\end{aligned}
\tag{1}
$$

The crossing factors $S(a)$ are defined by

$$
S(a) = (-1)^a\,\frac{\vartheta_1(4a\lambda)}{\vartheta_4(2a\lambda)} \tag{2}
$$

and $\vartheta_1(u)$, $\vartheta_4(u)$ are standard elliptic theta functions of the nome p

$$
\vartheta_1(u) = \vartheta_1(u,p) = 2p^{1/4}\sin u\prod_{n=1}^{\infty}\left(1-2p^{2n}\cos 2u+p^{4n}\right)\left(1-p^{2n}\right) \tag{3}
$$

$$
\vartheta_4(u) = \vartheta_4(u,p) = \prod_{n=1}^{\infty}\left(1-2p^{2n-1}\cos 2u+p^{4n-2}\right)\left(1-p^{2n}\right). \tag{4}
$$

Four different critical branches are defined by [8]

$$
\begin{array}{llll}
\text{Branch 1} & 0 < u < 3\lambda & \lambda = \dfrac{\pi}{4}\dfrac{L}{L+1} & L = 2, 3, \ldots \\
\text{Branch 2} & 0 < u < 3\lambda & \lambda = \dfrac{\pi}{4}\dfrac{L+2}{L+1} & L = 3, 4, \ldots \\
\text{Branch 3} & -\pi + 3\lambda < u < 0 & \lambda = \dfrac{\pi}{4}\dfrac{L+2}{L+1} & L = 3, 4, \ldots \\
\text{Branch 4} & -\pi + 3\lambda < u < 0 & \lambda = \dfrac{\pi}{4}\dfrac{L}{L+1} & L = 2, 3, \ldots .
\end{array}
\tag{5}
$$

This yields eight separate regimes, according to the sign of p. The magnetic Ising point occurs in regime 2 with $\lambda = \frac{5}{16}\pi$.

The integrable boundary weights are represented by a triangular form with three spins [1–3]. For the dilute models, in accordance with the adjacency condition of the model, we can define

$$
K\left(a \;\begin{matrix} c \\ b \end{matrix} \middle|\, u \right) = 0 \qquad \text{unless } |a-b| = 0, 1 \text{ and } |a-c| = 0, 1. \tag{6}
$$

These are to satisfy the boundary version of the star–triangle equation (the reflection equation) [14, 2, 3]

$$
\begin{aligned}
&\sum_{f,g} W\left(\begin{matrix} a & b \\ g & c \end{matrix} \middle|\, u-v \right) K\left(g \;\begin{matrix} c \\ f \end{matrix} \middle|\, u \right) W\left(\begin{matrix} a & g \\ d & f \end{matrix} \middle|\, u+v \right) K\left(d \;\begin{matrix} f \\ e \end{matrix} \middle|\, v \right) \\
&\qquad = \sum_{f,g} K\left(b \;\begin{matrix} c \\ f \end{matrix} \middle|\, v \right) W\left(\begin{matrix} a & b \\ g & f \end{matrix} \middle|\, u+v \right) K\left(g \;\begin{matrix} f \\ e \end{matrix} \middle|\, u \right) W\left(\begin{matrix} a & g \\ d & e \end{matrix} \middle|\, u-v \right)
\end{aligned}
\tag{7}
$$

ensuring the commutativity of the transfer matrix $\mathbf{T}(u)$ defined by the elements

$$
\begin{aligned}
\langle \boldsymbol{a} | \mathbf{T}(u) | \boldsymbol{b} \rangle = &\sum_{\{c_0,\ldots,c_N\}} K_+\left(\begin{matrix} a_0 \\ b_0 \end{matrix} \; c_0 \middle|\, u \right) \left[\prod_{k=0}^{N-1} W\left(\begin{matrix} c_k & c_{k+1} \\ b_k & b_{k+1} \end{matrix} \middle|\, u \right) \right. \\
&\left. \times W\left(\begin{matrix} c_k & a_k \\ c_{k+1} & a_{k+1} \end{matrix} \middle|\, u \right) \right] K_-\left(c_N \;\begin{matrix} a_N \\ b_N \end{matrix} \middle|\, u \right)
\end{aligned}
\tag{8}
$$

where $\boldsymbol{a} = \{a_0, a_1, \ldots, a_N\}$ and $\boldsymbol{b} = \{b_0, b_1, \ldots, b_N\}$ and

$$
K_-\left(a \;\begin{matrix} c \\ b \end{matrix} \middle|\, u \right) = K\left(a \;\begin{matrix} c \\ b \end{matrix} \middle|\, u \right) \tag{9}
$$

$$
K_+\left(\begin{matrix} c \\ b \end{matrix} \; a \middle|\, u \right) = \sqrt{\frac{S(a)^2}{S(b)S(c)}}\, K\left(a \;\begin{matrix} c \\ b \end{matrix} \middle|\, 3\lambda - u \right). \tag{10}
$$

Note that this formulation of the reflection equation and the transfer matrix differs from that given in [1]. The above formulation does not incorporate crossing symmetry of the bulk weights and is applicable to $A_n^{(1)}$, $n \geqslant 2$ [15] for which a crossing symmetry does not exist.

Another important relation is the boundary crossing relation, which here reads

$$\sum_c \sqrt{\frac{S(c)}{S(a)}} W\left(\begin{matrix} d & c \\ a & b \end{matrix} \,\middle|\, 2u+3\lambda\right) K\left(c \;\begin{matrix} d \\ b \end{matrix} \,\middle|\, u+3\lambda\right)$$

$$= \frac{\vartheta_1(2u+5\lambda)\vartheta_1(2u+6\lambda)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} K\left(a \;\begin{matrix} d \\ b \end{matrix} \,\middle|\, -u\right) . \tag{11}$$

It can easily be seen that the reflection equation reduces to five non-trivial equations with three distinct forms for diagonal boundary weights. These three distinct forms correspond to the generic equations in the higher rank $B_n^{(1)}$, $A_n^{(2)}$ cases, which have been solved in [15]. The diagonal solution we find is

$$K\left(a\pm 1 \;\begin{matrix} a \\ a \end{matrix} \,\middle|\, u\right) = \frac{\vartheta_4(\pm 2a\lambda+\xi-u)}{\vartheta_4(\pm 2a\lambda+\xi+u)}\, g_a(u) \tag{12}$$

$$K\left(a \;\begin{matrix} a \\ a \end{matrix} \,\middle|\, u\right) = \frac{\vartheta_1(-\lambda+\xi-u)}{\vartheta_1(-\lambda+\xi+u)}\, g_a(u) \tag{13}$$

where

$$g_a(u) = \vartheta_1(\xi+u)\vartheta_1(-\lambda+\xi+u)\vartheta_4(2a\lambda+\xi+u)\vartheta_4(-2a\lambda+\xi+u). \tag{14}$$

Here the parameter ξ takes the values $\xi = -\lambda/2 \bmod (\ell\pi/2 + m\pi\tau/2)$ where ℓ and m are integers and $g_a(u)$ has been fixed by crossing symmetry, otherwise $g_a(u)$ may be taken arbitrarily. The freedom in ξ due to the double periodicity of the elliptic functions gives four distinct solutions. All of these satisfy crossing symmetry up to an exponential factor. As for the bulk weights, height reversal symmetry is broken by the boundary weights for L odd, ensuring that the nome p can again be regarded as a magnetic field.

At criticality the known K-matrix solutions for the $A_2^{(2)}$ vertex model [18] are recovered via the usual passage from face weights to vertex weights. The integrable boundary weights of the dilute O(n) loop model [19] are also recovered in this limit after transforming to the diagonal orientation [5].

The fusion procedure has been applied to the dilute A_L models, resulting in the construction of both $su(2)$ [16] and $su(3)$ [17] fused face weights from the face weights given in (1). The $su(2)$ fusion rule provides the functional relation [16]

$$\mathbf{T}(u)\mathbf{T}(u+3\lambda) = \mathbf{f}(u) + \mathbf{T}^{(2)}(u) \tag{15}$$

where $\mathbf{T}^{(2)}(u)$ is the transfer matrix of the fused model with fusion level 2. For periodic boundary conditions the matrix function $\mathbf{f}(u)$ is given by

$$\mathbf{f}(u) = [\rho(u)]^N \mathbf{I} \tag{16}$$

where $\mathbf{I}$ is the identity matrix and

$$\rho(u) = \frac{\vartheta_1(2\lambda-u)\vartheta_1(3\lambda-u)\vartheta_1(2\lambda+u)\vartheta_1(3\lambda+u)}{\vartheta_1^2(2\lambda)\vartheta_1^2(3\lambda)} . \tag{17}$$

The fusion procedure can be carried out in a similar manner for the open boundary system, as has been done already for a number of IRF models [1, 2, 5, 6]. It is necessary to fuse the boundary face weights (12), (13). We find the matrix function $\mathbf{f}(u)$ to be diagonal with element (c, d) given by

$$f(u)_{c,d} = \omega_c^-(u)\omega_d^+(u)\rho^{2N}(u)/\rho(2u) \tag{18}$$

where the boundaries contribute the factors $\omega_c^-(u)$ and $\omega_d^+(u)$, with

$$\omega_c^-(u) = \sum_b \sqrt{\frac{S(b)}{S(c)}}\; W\left(\begin{matrix} c & b \\ c & c \end{matrix} \,\middle|\, 2u+3\lambda\right) K_-\left(b \begin{matrix} c \\ c \end{matrix} \,\middle|\, u+3\lambda\right) K_-\left(c \begin{matrix} c \\ c \end{matrix} \,\middle|\, u\right) \tag{19}$$

$$\omega_d^+(u) = \sum_b \sqrt{\frac{S(d)}{S(b)}}\; W\left(\begin{matrix} d & d \\ b & d \end{matrix} \,\middle|\, 3\lambda-2u\right) K_+\left(\begin{matrix} d \\ d \end{matrix}\; d \,\middle|\, u+3\lambda\right) K_+\left(\begin{matrix} d \\ d \end{matrix}\; b \,\middle|\, u\right). \tag{20}$$

Height c (d) is located on the right (left) boundary.

We now turn to the calculation of the magnetic surface exponents δ_s from the surface free energy. The finite-size corrections to the transfer matrix $\mathbf{T}(u)$ are contained in $\mathbf{T}^{(2)}(u)$. According to the spirit of [2, 4, 5, 6] the boundary crossing unitarity relation

$$T(u)T(u+3\lambda) = f(u)_{c,d} \tag{21}$$

for the eigenvalues is sufficient to determine both the bulk and surface free energies. The two contributions can be separated out by setting $T(u) = T_\mathrm{b}(u)T_\mathrm{s}(u)$ and defining $T_\mathrm{b} = \kappa_\mathrm{b}^{2N}$ and $T_\mathrm{s} = \kappa_\mathrm{s}$. The free energies per site then follow as $f_\mathrm{b}(u) = -\log\kappa_\mathrm{b}(u)$ and $f_\mathrm{s}(u) = -\log\kappa_\mathrm{s}(u)$. Thus for the bulk contribution we have

$$\kappa_\mathrm{b}(u)\kappa_\mathrm{b}(u+3\lambda) = \rho(u)\,. \tag{22}$$

This relation has already been used to determine the bulk free energy [8, 12] via the inversion relation method [13, 20]. The critical behaviour as $p \to 0$, obtained by use of the Poisson summation formula [13], is [8, 12]

$$f_\mathrm{b} \sim \begin{cases} p^{1+1/\delta_\mathrm{b}} & L \text{ odd} \\ p^{2-\alpha_\mathrm{b}} & L \text{ even} \end{cases} \tag{23}$$

where the values of δ_b and α_b are listed for the different regimes in table 1.

Table 1. Magnetic and thermal critical bulk and surface exponents of the dilute A_L models.

Regime	1	2	3	4	
δ_b	$\frac{3L}{L+4}$	$\frac{3(L+2)}{L-2}$	$\frac{L-2}{3(L+2)}$	$\frac{L+4}{3L}$	L odd
δ_s	$\frac{-3L}{L-2}$	$\frac{-3(L+2)}{L+4}$	$\frac{L+4}{L-2}$	$\frac{L+4}{L-2}$	L odd
α_b	$\frac{2(L-2)}{3L}$	$\frac{2(L+4)}{3(L+2)}$	$\frac{-2(L+4)}{L-2}$	$\frac{-2(L-2)}{L+4}$	L even
α_s	$\frac{2(2L-1)}{3L}$	$\frac{2(2L+5)}{3(L+2)}$	$\frac{-6}{L-2}$	$\frac{6}{L+4}$	L even

To determine the surface exponents δ_s and α_s it is sufficient to consider only the relevant contribution to the surface free energy, which does not involve the explicit form of the boundary weights. By making use of the boundary crossing relation and after taking a convenient normalization, the appropriate contribution is obtained by solving

$$\kappa_\mathrm{s}(u)\kappa_\mathrm{s}(u+3\lambda) = \frac{\vartheta_1(5\lambda-2u)\vartheta_1(6\lambda-2u)\vartheta_1(5\lambda+2u)\vartheta_1(6\lambda+2u)}{\vartheta_1^2(5\lambda)\vartheta_1^2(6\lambda)}\,. \tag{24}$$

We solve this relation under the same analyticity assumptions as for the bulk calculation [8, 12]. Specifically, we set $p = \mathrm{e}^{-\epsilon}$, make the relevant conjugate modulus transformations, Laurent expand $\log \kappa_s(u)$ in powers of $\exp(-2\pi u/\epsilon)$ and match coefficients in (24). In regimes 1 and 2 we obtain

$$f_s = -4 \sum_{k=1}^{\infty} \frac{\cosh(\pi\lambda k/\epsilon)\ \cosh[(11\lambda-\pi)\pi k/\epsilon]\ \sinh(2\pi u k/\epsilon)\ \sinh[2(3\lambda-u)\pi k/\epsilon]}{k\ \sinh(\pi^2 k/\epsilon)\ \cosh(6\pi\lambda k/\epsilon)}\ . \tag{25}$$

For regimes 3 and 4 the LHS of relation (24) needs to be modified to $\kappa_s(u)\kappa_s(u+3\lambda-\pi)$ for the appropriate analyticity strip, with the end result

$$f_s = 4 \sum_{k=1}^{\infty} \frac{\cosh(\pi\lambda k/\epsilon)\ \cosh[(11\lambda-\pi)\pi k/\epsilon]\ \sinh(2\pi u k/\epsilon)\ \sinh[2(\pi-3\lambda+u)\pi k/\epsilon]}{k\ \sinh(\pi^2 k/\epsilon)\ \cosh[2(\pi-3\lambda)k/\epsilon]}\ . \tag{26}$$

Application of the Poisson summation formula yields

$$f_s \sim \begin{cases} p^{1+1/\delta_s} & L \text{ odd} \\ p^{2-\alpha_s} & L \text{ even} \end{cases} \tag{27}$$

as $p \to 0$, where the values of δ_s and α_s are listed for the various regimes alongside the bulk values in table 1. As in the bulk case, these exponents are magnetic for L odd and thermal for L even.

For $L = 3$ in regime 2 we obtain the value $\delta_s = -\frac{15}{7}$. This result is in agreement with the prediction for the two-dimensional Ising model in a magnetic field using the scaling relations between bulk and surface exponents [21, 22, 23, 11]†. The negative value indicates that the surface magnetization diverges towards infinity as the magnetic field goes to zero. Thus a surface magnetization in zero field does not exist. Similar behaviour is known to occur in the spherical model [11]. As expected, the Ising specific heat exponents $\alpha_b = 0$ and $\alpha_s = 1$ are recovered in regimes 1 and 4 for $L = 2$.

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References

[1] Behrend R E, Pearce P A and O'Brien D L 1995 Interaction-round-a-face models with with fixed boundary conditions: The ABF fusion hierarchy *Preprint* hep-th/9507118
[2] Zhou Y-K 1996 *Nucl. Phys.* B in press
[3] Ahn C and Koo W M 1995 Boundary Yang–Baxter equation in the RSOS representation *Preprint* hep-th/9508080
[4] Batchelor M T and Zhou Y-K 1996 *Phys. Rev. Lett.* **76** 14
[5] Zhou Y-K and Batchelor M T Surface critical phenomena in interaction-round-a-face models *ANU preprint* MRR-070-95, cond-mat/9511008
[6] Zhou Y-K and Batchelor M T Exact solution and surface critical behaviour of open cyclic SOS lattice models *ANU preprint* MRR-073-95
[7] Andrews G E, Baxter R J and Forrester P J 1984 *J. Stat. Phys.* **35** 193
[8] Warnaar S O, Nienhuis B and Seaton K A 1992 *Phys. Rev. Lett.* **69** 710
Warnaar S O, Nienhuis B and Seaton K A 1993 *Int. J. Mod. Phys.* **7B** 3727
[9] Roche Ph 1992 *Phys. Lett.* **285B** 49
[10] McCoy B M and Wu T T 1967 *Phys. Rev.* **162** 436

† Actually there is a misprint in the tabulated result for δ_s in Binder's review [11].

[11] Binder K 1983 *Phase Transitions and Critical Phenomena* vol 8, ed C Domb and J L Lebowitz (New York: Academic)

[12] Warnaar S O, Pearce P A, Seaton K A and Nienhuis B 1994 *J. Stat. Phys.* **74** 469

[13] Baxter R J 1982 *Exactly Solved Lattice Models in Statistical Mechanics* (New York: Academic)

[14] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375

[15] Batchelor M T, Fridkin V, Kuniba A and Zhou Y K 1995 Solutions of the reflection equation for face and vertex models associated with $A_n^{(1)} B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ and $A_n^{(2)}$, in preparation

[16] Zhou Y K 1995 $SU(2)$ Hierarchies of dilute lattice models *Preprint* hep-th/9506107

[17] Zhou Y K, Pearce P A and Grimm U 1995 *Physica* **222A** 261

[18] Mezincescu L and Nepomechie R I 1991 *Int. J. Mod. Phys.* **7A** 5231

[19] Yung C M and Batchelor M T 1995 *Nucl. Phys.* B **435** 430

[20] Baxter R J 1982 *J. Stat. Phys.* **28** 1

[21] Binder K and Hohenberg P C 1972 *Phys. Rev.* B **6** 3461
Binder K and Hohenberg P C 1974 *Phys. Rev.* B **9** 2194

[22] Barber M N 1973 *Phys. Rev.* B **8** 407

[23] Fisher M E 1973 *J. Vac. Sci. Technol.* **10** 665